

General Linear Quadratic Optimal Stochastic Control Problem Driven by a Brownian Motion and a Poisson Random Martingale Measure with Random Coefficients *

Qingxin Meng

School of Mathematical Sciences, Fudan University, Shanghai 200433, China

Email: 071018034@fudan.edu.cn

Abstract

Consider the minimization of the following quadratic functional

$$J(u) = E \int_0^T [\langle Q_t X_t, X_t \rangle dt + \langle N_t u_t, u_t \rangle] dt + E \langle M X_T, X_T \rangle,$$

where X is the strong solution to the linear state equation driven by a multidimensional Brownian motion W and a Poisson random martingale measure $\tilde{\mu}(d\theta, dt)$

$$\begin{cases} dX_t = (A_t X_t + B_t u_t) dt + \sum_{i=1}^d (C_t^i X_t + D_t^i u_t) dW_t^i \\ \quad + \int_Z (E_t(\theta) X_{t-} + F_t(\theta) u_t) \tilde{\mu}(d\theta, dt), \\ x_0 = x. \end{cases}$$

Here u is a square integrable adapted control process. The problem is conventionally called the stochastic linear quadratic (LQ in short form) optimal control problem. This paper is concerned the following general case: the coefficients $A, B, C^i, D^i, E, F, Q, N$ and M are allowed to be predictable processes or random matrices. Associated with this LQ problem, the corresponding Riccati equation is a multidimensional backward stochastic differential equation driven by the Brownian motion W and the Poisson random martingale measure $\tilde{\mu}(d\theta, dt)$ (see (5.9)). The backward stochastic Riccati differential equation with jumps will be abbreviated as BSRDEJ. The generator of BSRDEJ is highly nonlinear in the three unknown variables K, L and H (see (5.9)).

In the paper, we will establish the connections of the multidimensional BSRDEJ to the stochastic LQ problem and to the associated Hamilton systems. By the

*This work is partially supported by the National Basic Research Program of China (973 Program) (Grant No.2007CB814904), the National Natural Science Foundation of China (Grants No.10325101, 11071069), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No.20090071120002) and the Innovation Team Foundation of the Department of Education of Zhejiang Province (Grant No.T200924)..

connections, we show the optimal control have the state feedback representation. Moreover, we will show the existence and uniqueness result of the multidimensional BSRDEJ for the case where the generator is bounded linear dependence with respect to the unknown martingale term L and H .

Keywords: Poisson random martingale measure, Linear quadratic optimal stochastic control, Random coefficients Dynamic programming, Itô-ventzell formula, Riccati equation, Backward stochastic differential equations, Stochastic Hamilton system

1 Introduction

Linear Quadratic (LQ in short form) optimal control problem is a problem where the system dynamics are linear in state and control variables and the cost functional is quadratic in the two variables. It is well known that LQ problem is one of the most important classes of optimal control problem, and the solution of this problem has had a profound impact on many engineering applications and mathematical finance.

The very first attempt in tracking deterministic LQ problem was made by Bellman, Glicksberg and Gross [2] in 1958. However, Kalman[11] has been widely credited for his pioneering work published in 1960, in solving the problem in a linear state feedback control form. Since then, the problem has been extensively studied and developed in major research field in control theory. Extension to stochastic LQ control was first carried out by Wonham [20]. Bismut [3] performed a detailed analysis for stochastic LQ control with random coefficients. With the joint effort of many researchers in the last 50 years, there has been an enormously rich theory on LQ control, deterministic and stochastic alike (see [17],[4],[6],[7],[14],[10],[22]).

One of the elegant features of the LQ theory is that it is able to give in explicit forms the optimal state feedback control and the optimal cost value through the celebrated Riccati equation. Associated with deterministic LQ problem or stochastic LQ problem with deterministic coefficients, the corresponding Riccati equation is backward deterministic ordinary differential equation. For the deterministic Riccati equation was essentially solved by Wonham [20] by applying Bellman's principle of quasilinearization (see Bellman[1]) and a monotone convergence result of symmetric matrices.

But associated with stochastic LQ problem with random coefficients, the corresponding Riccati equation is a highly nonlinear backward stochastic differential equations where the generator depends on the unknown variable in quadratic way. This sort of Riccati equation is called backward stochastic Riccati equation (BSRDE in short form). The interest of proving existence and uniqueness results for such a class of equations was first addressed by Bismut in [3]. It was clear from the beginning that to study those highly nonlinear backward stochastic differential equation (BSDE in short form) was already a challenging task and turned out to become a long-standing problem. The difficulty comes essentially from the fact that, in its general formulation, the BSRDE involves quadratic terms in both the unknowns (in particular in the so-called martingale term). Moreover the nonlinearity can be well defined only in a subset of the space of nonnegative matrices (where the equation naturally exists).

For the special case that the generator of BSRDE depends on the unknowns martingale term only in linear way, Bismut[3] obtained the existence and uniqueness result by

constructing a contraction mapping and the using a fixed point theorem and in 1992, Peng[16] also gave a nice treatment on the proof of existence and uniqueness by using Bellman's linearization and a monotone convergence result of symmetric matrices-a generalization of Wonham's approach to the random situation. Later Kohlmann and Tang have made some progress towards solving the open problem. See [12, 13] and the references therein. However it is still far from the complete solution. Until 2003, by the methods of stochastic flows, Tang [18] solved the long standing open problem of the proof of the existence and uniqueness of the solution of the BSRDE in the general case corresponding to a linear quadratic problem with random coefficients and state-and control-dependent noise. In this work[18], Tang provides a rigorous derivation between the Riccati equation and the stochastic Hamilton system as two different but equivalent tools for the stochastic LQ problem.

For the discontinuous LQ problem, in 2003, Wu and Wang [21] discussed the stochastic LQ problem with the system driven by Brownian motion and Poisson jumps and obtain the existence and uniqueness result of a class of deterministic Riccati equation. And in 2008, Hu and Øksendal [9] studied the stochastic LQ problem for the one-dimensional case with Poisson jumps and random coefficients under partial information, and the main result is to show the optimal control has state feedback representation by an one-dimensional BSRDE with jumps in view of the technique of completing squares. But in [9], the author did not discussed the existence and uniqueness of the solution to BSRDE with jumps.

So for the LQ problem with jumps, it is still far from the complete solution. The main purpose of this paper is to discuss detailed the stochastic LQ control problem with random coefficients where the linear system is a multidimensional stochastic differential equation driven by a multidimensional Brownian motion and a Poisson random martingale measure. In the paper, we will establish the connections of the multidimensional Backward stochastic Riccati equation with jumps (BSRDEJ in short form) to the stochastic LQ problem and to the associated Hamilton systems. By the connections, we show the optimal control have the state feedback representation. Moreover, we will show the existence and uniqueness result of the multidimensional BSRDEJ for the case where the generator is bounded linear dependence with respect to the unknowns martingale term.

The rest of the paper is organized as follows. In section 2 we introduce useful notation and some existing results on stochastic differential equations (SDEs in short form) and BSDEs driven by Poisson random martingale measure. In section 3, we state the stochastic LQ problem we study, give needed assumptions and prove some preliminary property on the functional cost. Moreover, we have showed the stochastic LQ problem with jumps has a unique optimal control. In section 4, we establish the dual characterization of the optimal control by stochastic Hamilton system. In section 5, we will present the main results. In this section, we will introduce BSRDEJ and establish the link with the stochastic Hamilton system with jumps, then show the optimal control of the stochastic LQ problem has state feedback representation. In the end, we will focus on discussing the existence and uniqueness of the solution to BSRDEJ.

2 Notation and Preliminaries

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space. In this probability space, there is a d -dimensional standard Brownian motion $\{W_t\}_{t \geq 0}$ and a stationary Poisson point process $\{\eta_t\}_{t \geq 0}$ defined on a fixed nonempty measurable subset Z of R^1 . We denote by $\mu(de, dt)$ the counting measure induced by $\{\eta_t\}_{t \geq 0}$ and by $\nu(d\theta)$ the corresponding characteristic measure. Furthermore, We assume that $\nu(Z) < \infty$. Then the compensate random martingale measure is denoted by $\tilde{\mu}(d\theta, dt) := \mu(d\theta, dt) - \nu(d\theta)dt$. We can assume that $\{\mathcal{F}_t\}_{t \geq 0}$ is the P-augmentation of the natural filtration generated by $\{W_t\}_{t \geq 0}$ and $\{\eta_t\}_{t \geq 0}$. Denote by \mathcal{P} the predictable sub- σ field of $\mathcal{B}([0, T]) \times \mathcal{F}$, then we introduce the following notation used throughout this paper.

- H : a Hilbert space with norm $\|\cdot\|_H$.
- $\langle \alpha, \beta \rangle$: the inner product in $\mathbb{R}^n, \forall \alpha, \beta \in \mathbb{R}^n$.
- $|\alpha| = \sqrt{\langle \alpha, \alpha \rangle}$: the norm of $\mathbb{R}^n, \forall \alpha \in \mathbb{R}^n$.
- $\langle A, B \rangle = \text{tr}(AB^T)$: the inner product in $\mathbb{R}^{n \times m}, \forall A, B \in \mathbb{R}^{n \times m}$.
- $|A| = \sqrt{\text{tr}(AA^*)}$: the norm of $\mathbb{R}^{n \times m}, \forall A \in \mathbb{R}^{n \times m}$. Here we denote by A^* , the transpose of a matrix A .
- S^n : the set of all $n \times n$ symmetric matrices.
- S_+^n : the subset of all non-negative definite matrices of S^n .
- $(S^n)^l := \underbrace{S^n \times \cdots \times S^n}_l$.
- $S_{\mathcal{F}}^2(0, T; H)$: the space of all H -valued and \mathcal{F}_t -adapted càdlàg processes $f = \{f(t, \omega), (t, \omega) \in [0, T] \times \Omega\}$ satisfying

$$\|f\|_{S_{\mathcal{F}}^2(0, T; H)} \triangleq \sqrt{E \sup_{0 \leq t \leq T} \|f(t)\|_H^2} < +\infty.$$

- $\mathcal{L}_{\mathcal{F}}^2(0, T; H)$: the space of all H -valued and \mathcal{F}_t -adapted processes $f = \{f(t, \omega), (t, \omega) \in [0, T] \times \Omega\}$ satisfying

$$\|f\|_{\mathcal{L}_{\mathcal{F}}^2(0, T; H)} \triangleq \sqrt{E \int_0^T \|f(t)\|_H^2 dt} < \infty.$$

- $\mathcal{L}^{\nu, 2}(Z; H)$: the space of H -valued measurable functions $r = \{r(\theta), \theta \in Z\}$ defined on the measure space $(Z, \mathcal{B}(Z); \nu)$ satisfying

$$\|r\|_{\mathcal{L}^{\nu, 2}(Z; H)} \triangleq \sqrt{\int_Z \|r(\theta)\|_H^2 \nu(d\theta)} < \infty.$$

- $\mathcal{L}_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; H)$: the space of $\mathcal{L}^{\nu, 2}(Z; H)$ -valued and \mathcal{F}_t -predictable processes $r = \{r(t, \omega, \theta), (t, \omega) \in [0, T] \times \Omega \times Z\}$ satisfying

$$\|r\|_{\mathcal{L}_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; H)} \triangleq \sqrt{E \iint_{Z \times (0, T]} \|r(t, \theta)\|_H^2 \nu(d\theta) dt} < \infty.$$

- $L^2(\Omega, \mathcal{F}, P; H)$: the space of all H -valued random variables ξ on (Ω, \mathcal{F}, P) satisfying

$$\|\xi\|_{L^2(\Omega, \mathcal{F}, P; H)} \triangleq E\|\xi\|_H^2 < \infty.$$

Now we give two preliminary lemmas about SDE and BSDE driven by the d -dimensional Brownian motion W_t and the Poisson random martingale measure $\tilde{\mu}(d\theta, dt)$. which will often been used in this paper.

Lemma 2.1. *Let a an \mathcal{F}_0 -measurable random variable and*

$$\begin{aligned} b : [0, T] \times \Omega \times R^n &\longrightarrow R^n, \\ \sigma : [0, T] \times \Omega \times R^n &\longrightarrow R^{n \times m}, \\ \pi : [0, T] \times \Omega \times Z \times R^n &\longrightarrow R^n \end{aligned}$$

are given mappings satisfying the following assumptions

(i) b, σ and π are measurable with respect to $\mathcal{P} \times \mathcal{B}(R^n) / \mathcal{B}(R^n)$, $\mathcal{P} \times \mathcal{B}(R^n) / \mathcal{B}(R^{n \times d})$, $\mathcal{P} \times \mathcal{B}(Z) \times \mathcal{B}(R^n) / \mathcal{B}(R^n)$ respectively.

(ii) $b(\cdot, 0) \in \mathcal{L}_{\mathcal{F}}^2(0, T; R^n)$; $\sigma(\cdot, 0) \in \mathcal{L}_{\mathcal{F}}^2(0, T; R^{n \times d})$; $\pi(\cdot, \cdot, 0) \in \mathcal{L}_{\mathcal{F}}^{\nu, 2}([0, T] \times Z, R^n)$.

(iii) b, σ and π are uniformly Lipschitz continuous w.r.t. x , i.e. there exists a constant $C > 0$ s.t. for all $(t, x, \bar{x}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ and a.s. $\omega \in \Omega$,

$$\begin{aligned} &|b(t, x) - b(t, \bar{x})|^2 + |\sigma(t, x) - \sigma(t, \bar{x})|^2 \\ &+ \int_Z |\pi(t, \theta, x) - \pi(t, \theta, \bar{x})|^2 \nu(d\theta) \leq C|x - \bar{x}|^2. \end{aligned} \quad (2.1)$$

Then the SDE with jumps

$$X_t = a + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s + \iint_{Z \times (0, T]} \pi(s, \theta, X_{s-}) \tilde{\mu}(d\theta, ds) \quad (2.2)$$

has a unique solution $X \in S_{\mathcal{F}}^2(0, T; R^n)$. Moreover, the following a priori estimate holds

$$\begin{aligned} E \sup_{0 \leq t \leq T} |X_t|^2 &\leq K \left[E \int_0^T |b(t, 0)|^2 dt + E \int_0^T |\sigma(t, 0)|^2 dt \right. \\ &\quad \left. + E \iint_{Z \times (0, T]} |\pi(t, \theta, 0)|^2 \nu(d\theta) dt + E|a|^2 \right], \end{aligned} \quad (2.3)$$

where K is a positive constant depending only on Lipschitz constant C and T .

Lemma 2.2. *Let ξ an \mathcal{F}_T -measurable random variable and*

$$f : [0, T] \times \Omega \times R^n \times R^{n \times d} \times \mathcal{L}^{\nu, 2}(Z; R^n) \longrightarrow R^n \quad (2.4)$$

is a given mapping satisfying the following assumptions

(i) f is measurable with respect to $\mathcal{P} \times \mathcal{B}(R^n) \times \mathcal{B}(R^{n \times d}) \times \mathcal{B}(\mathcal{L}^{\nu, 2}(Z; R^n)) / \mathcal{B}(R^n)$

(ii) $f(\cdot, 0, 0, 0) \in \mathcal{L}_{\mathcal{F}}^2(0, T; R^n)$; $\xi \in \mathcal{L}^2(\Omega, \mathcal{F}, P; R^n)$.

(iii) f is uniformly Lipschitz continuous w.r.t. (y, q, r) , i.e. there exists a constant $C > 0$

s.t. for all $(t, y, q, r, \bar{y}, \bar{q}, \bar{r}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathcal{L}^{\nu, 2}(Z; \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathcal{L}^{\nu, 2}(Z; \mathbb{R}^n)$ and a.s. $\omega \in \Omega$,

$$\begin{aligned} & |f(t, y, q, r) - f(t, \bar{y}, \bar{q}, \bar{r})|^2 \\ & \leq C \left[|y - \bar{y}|^2 + |q - \bar{q}|^2 + \int_Z |r(\theta) - \bar{r}(\theta)|^2 \nu(d\theta) \right]. \end{aligned} \quad (2.5)$$

Then the BSDE with jumps

$$Y_t = \xi + \int_t^T f(s, Y_s, Q_s, R_s) ds - \int_t^T Q_s dW_s - \iint_{Z \times (t, T]} R_s(\theta) \tilde{\mu}(d\theta, ds) \quad (2.6)$$

has a unique solution

$$(Y, Q, R) \in S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times \mathcal{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^{n \times d}) \times \mathcal{L}_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; \mathbb{R}^n).$$

Moreover, we have the following a priori estimate

$$\begin{aligned} & E \sup_{0 \leq t \leq T} |Y_t|^2 + E \int_0^T |Q_t|^2 dt + E \iint_{Z \times (0, T]} |R_t(\theta)|^2 \nu(d\theta) dt \\ & \leq K \left[E \int_0^T |f(t, 0, 0, 0)|^2 dt + E |\xi|^2 \right], \end{aligned} \quad (2.7)$$

where K is a positive constant depending only on C and T .

Particularly, if

$$M := \sup_{\omega \in \Omega} \left[\int_0^T |f(t, \omega, 0, 0, 0)|^2 dt + |\xi(\omega)|^2 \right] < \infty, \quad (2.8)$$

then for all $t \in [0, T]$ and a.s., we have

$$|Y_t|^2 < M \cdot e^{KT}, \quad (2.9)$$

where K is a positive constant depending only on Lipschitz constant C .

Proof. The proof of the existence and the uniqueness can be found in [19]. In the following we will only proof the estimate (2.9). As for the a priori estimate (2.7), it can be obtained similarly by Gronwall's inequality and Burkholder-Davis-Gundy inequality. In fact, for any given $0 \leq r \leq t \leq T$, applying Itô's formula to $|y_t|^2$ and taking conditional

expectation with respect to \mathcal{F}_r , we have

$$\begin{aligned}
& E^{\mathcal{F}_r} |Y_t|^2 + E^{\mathcal{F}_r} \int_t^T |Q_s|^2 ds + E^{\mathcal{F}_r} \iint_{Z \times (t, T]} |R_s(\theta)|^2 \nu(d\theta) ds \\
&= E^{\mathcal{F}_r} \int_t^T 2 \langle f(s, Y_s, Q_s, R_s), Y_s \rangle ds + E^{\mathcal{F}_r} |\xi|^2 \\
&\leq E^{\mathcal{F}_r} \int_t^T 2 |f(s, Y_s, Q_s, R_s)| |Y_s| ds + E^{\mathcal{F}_r} |\xi|^2 \\
&\leq E^{\mathcal{F}_r} \int_t^T 2 |f(s, Y_s, Q_s, R_s) - f(s, 0, 0, 0) + f(s, 0, 0, 0)| |Y_s| ds + E^{\mathcal{F}_r} |\xi|^2 \\
&\leq \frac{1}{2C} E^{\mathcal{F}_r} \int_t^T |f(s, Y_s, Q_s, R_s) - f(s, 0, 0, 0)|^2 ds + 2C E^{\mathcal{F}_r} \int_t^T |Y_s|^2 ds \\
&\quad + E^{\mathcal{F}_r} \int_t^T |f(s, 0, 0, 0)|^2 ds + E^{\mathcal{F}_r} \int_t^T |Y_s|^2 ds + E^{\mathcal{F}_r} |\xi|^2 \\
&\leq (2C + \frac{3}{2}) E^{\mathcal{F}_r} \int_t^T |Y_s|^2 ds + \frac{1}{2} E^{\mathcal{F}_r} \int_t^T |Q_s|^2 ds + \frac{1}{2} E^{\mathcal{F}_r} \iint_{Z \times (t, T]} |R_s(\theta)|^2 \nu(d\theta) ds \\
&\quad + E^{\mathcal{F}_r} \left[\int_t^T |f(s, 0, 0, 0)|^2 ds + \mathcal{F}_r |\xi|^2 \right],
\end{aligned} \tag{2.10}$$

where the Lipschitz condition (2.5) and the basic inequality $2ab \leq \beta a^2 + \frac{1}{\beta} b^2$, $\forall \beta > 0, a > 0, b > 0$ are used. Therefore, we have

$$\begin{aligned}
E^{\mathcal{F}_r} |Y_t|^2 &\leq E^{\mathcal{F}_r} \left[\int_0^T |f(s, 0, 0, 0)|^2 ds + |\xi|^2 \right] + (2C + \frac{3}{2}) E^{\mathcal{F}_r} \int_t^T |y_s|^2 ds \\
&\leq M + K \int_t^T E^{\mathcal{F}_r} |Y_s|^2 ds,
\end{aligned} \tag{2.11}$$

where we set $L = 2C + \frac{3}{2}$.

Consequently, applying Gronwall's inequality, we get

$$E^{\mathcal{F}_r} |Y_t|^2 \leq M e^{K(T-t)}, \quad 0 \leq r \leq t \leq T. \tag{2.12}$$

In the end, particularly taking $r = t$, we obtain the estimate (2.9)

□

3 Formulation of the problem and Elementary Results

Consider the following linear stochastic system derived by Brownian motion W_t and Poisson random measure $\tilde{\mu}(d\theta, dt)$

$$\begin{cases} dX_t = (A_t X_t + B_t u_t)dt + \sum_{i=1}^d (C_t^i X_t + D_t^i u_t) dW_t^i \\ \quad + \int_Z (E_t(\theta) X_{t-} + F_t(\theta) u_t) \tilde{\mu}(d\theta, dt), \\ x_0 = x. \end{cases} \quad (3.1)$$

The process u in (3.1) is our control process. An admissible control u is defined as a $\{\mathcal{F}_t, 0 \leq t \leq T\}$ -predictable process with values in R^m such that $E \int_0^T |u(t)|^2 dt < +\infty$. The set of all admissible control u is denoted by \mathcal{A} . Note that \mathcal{A} is a Hilbert space.

And for any admissible control $u \in \mathcal{A}$, we consider the following quadratic cost functional

$$J(u) = E \int_0^T [\langle Q_t X_t, X_t \rangle dt + \langle N_t u_t, u_t \rangle] dt + E \langle M X_T, X_T \rangle, \quad (3.2)$$

where X is the strong solution to the state equation (3.1).

Throughout this paper, we make the following assumptions on the coefficients $A, B, C^i, D^i, E, F, Q, N$ and M .

Assumption 3.1. The matrix processes $A : [0, T] \times \Omega \rightarrow R^{n \times n}, B : [0, T] \times \Omega \rightarrow R^{n \times m}, C^i : [0, T] \times \Omega \rightarrow R^{n \times n}, D^i : [0, T] \times \Omega \rightarrow R^{n \times m}, i = 1, 2, \dots, d; E : [0, T] \times \Omega \rightarrow \mathcal{L}^{v,2}(Z; R^{n \times n}), F : [0, T] \times \Omega \rightarrow \mathcal{L}^{v,2}(Z; R^{n \times m}); Q : [0, T] \times \Omega \rightarrow R^{n \times n}, N : [0, T] \times \Omega \rightarrow R^{m \times m}$; and the random matrix $M : \Omega \rightarrow R^{n \times n}$ are uniformly bounded and $\{\mathcal{F}_t, 0 \leq t \leq T\}$ -predictable or \mathcal{F}_T -measurable.

Assumption 3.2. The state weighting matrix process Q and the control weighting matrix process N are a.s. a.e. symmetric and nonnegative. The terminal state weighting random matrix M is a.s. symmetric and nonnegative. The control weighting matrix process N is a.s. a.e. uniformly positive, i.e. $N(t) \geq \delta I$ for some positive constant δ and almost all $(t, \omega) \in [0, T] \times \Omega$.

Under Assumption 3.1, from Lemma 2.1, the system (3.1) admits a unique solution strong solution, which will be denoted by $X^{(x,u)}$ or X if its dependence on admissible control u is clear from the context. Then we call X the state process corresponding to the control process u and $(u; X)$ the admissible pair. Furthermore, from Assumption 3.2 and the a priori estimate (2.3), it is easy to check that

$$|J(u)| < \infty.$$

Then we can pose the so-called linear quadratic (LQ) problem.

Problem 3.1. Find an admissible control \bar{u} such that

$$J(\bar{u}) = \inf_{u \in \mathcal{A}} J(u) \quad (3.3)$$

Any $\bar{u} \in \mathcal{A}$ satisfying the above is called an optimal control process of Problem 3.1 and the corresponding state process \bar{X} is called the corresponding optimal state process. We also refer to $(\bar{u}; \bar{X})$ as an optimal pair of Problem 3.1.

Lemma 3.2. *Under Assumptions 3.1-3.2, the cost functional J is strictly convex over \mathcal{A} . Moreover, $\lim_{\|u\|_{\mathcal{A}} \rightarrow \infty} J(u) = +\infty$*

Proof. Under Assumption 3.2, by the definition of cost functional J (see (3.2)), it is easy to check that J is a convex functional. Since the weighting matrix process N is uniformly strictly positive, we can conclude that J is strictly convex over \mathcal{A} . Moreover, in view of the nonnegative property of Q, M and the uniformly strictly positive property of N , we have

$$0 \geq J(u) \geq \delta E \int_0^T |u_t|^2 dt = \delta \|u\|_{\mathcal{A}}^2.$$

Therefore, $\lim_{\|u\|_{\mathcal{A}} \rightarrow \infty} J(u) = +\infty$. □

Lemma 3.3. *Under Assumptions 3.1-3.2, the cost functional J is Fréchet differentiable over \mathcal{A} . Moreover, the corresponding Fréchet derivative J' at any admissible control $u \in \mathcal{A}$ is given by*

$$\langle J'(u), v \rangle = 2E \int_0^T \left[\langle Q_t X_t^{(x,u)}, X_t^{(0,v)} \rangle + \langle N_t u_t, v_t \rangle \right] dt + 2E \langle M X_T^{(x,u)}, X_T^{(0,v)} \rangle, \quad \forall v \in \mathcal{A}, \quad (3.4)$$

where $X^{(0,v)}$ is the solution of the SDE (3.1) corresponding to the admissible control v and the initial value $X_0 = 0$, and $X^{(x,u)}$ is the state process corresponding to the control process u .

Proof. For $\forall u, v \in \mathcal{A}$, we define

$$\Delta J := J(u + v) - J(u) - 2E \int_0^T \left[\langle Q_t X_t^{(x,u)}, X_t^{(0,v)} \rangle + \langle N_t u_t, v_t \rangle \right] dt - 2E \langle M X_T^{(x,u)}, X_T^{(0,v)} \rangle,$$

Then from the definition of cost functional J (see(3.2)), we have

$$\begin{aligned} \Delta J &= E \int_0^T \left[\langle Q_t (X_t^{(x,u)} + X_t^{(0,v)}), X_t^{(x,u)} + X_t^{(0,v)} \rangle + \langle N_t (u_t + v_t), u_t + v_t \rangle \right] dt \\ &\quad + E \langle M (X_T^{(x,u)} + X_T^{(0,v)}), X_T^{(x,u)} + X_T^{(0,v)} \rangle - E \int_0^T \left[\langle Q_t X_t^{(x,u)}, X_t^{(x,u)} \rangle \right. \\ &\quad \left. + \langle N_t u_t, u_t \rangle \right] dt - E \langle M X_T^{(x,u)}, X_T^{(x,u)} \rangle - 2E \int_0^T \left[\langle N_t X_t^{(x,u)}, X_t^{(0,v)} \rangle \right. \\ &\quad \left. + \langle Q_t u_t, v_t \rangle \right] dt - 2E \langle M X_T^{(x,u)}, X_T^{(0,v)} \rangle \\ &= 2E \int_0^T \left[\langle Q_t X_t^{(0,v)}, X_t^{(0,v)} \rangle + \langle N_t v_t, v_t \rangle \right] dt + 2E \langle M X_T^{(0,v)}, X_T^{(0,v)} \rangle. \end{aligned} \quad (3.5)$$

Then it follows from Assumptions 3.1 and the a priori estimate (2.3) that

$$|\Delta J| \leq KE \int_0^T |v_t|^2 dt = K \|v\|_{\mathcal{A}}^2$$

Consequently, we deduce that

$$\lim_{\|v\|_{\mathcal{A}} \rightarrow 0} \frac{|\Delta J|}{\|v\|_{\mathcal{A}}} = 0,$$

which implies that J is Fréchet differentiable and its Fréchet derivative J' is given by (3.4). \square

Theorem 3.4. *Under Assumptions 3.1-3.2, Problem 3.1 has a unique optimal control u .*

Proof. In view of the fact that the cost functional J is Fréchet differentiable, strictly convex and $\lim_{\|u\|_{\mathcal{A}} \rightarrow \infty} J(u) = +\infty$, the existence and uniqueness of the optimal control can be directly obtained by Proposition 2.1.2 in [8]. \square

Theorem 3.5. *Under Assumptions 3.1-3.2, a necessary and sufficient conditions for an admissible control $u \in \mathcal{A}$ to be an optimal control of Problem 3.1 is for any admissible control $v \in \mathcal{A}$,*

$$\langle J'(u), v - u \rangle = 0. \quad (3.6)$$

Proof. Since the cost functional J is Fréchet differentiable and strictly convex, according to Proposition 2.2.1 in [8], we conclude that a necessary and sufficient conditions for an admissible control $u \in \mathcal{A}$ to be an optimal control of Problem 3.1 is for any admissible control $v \in \mathcal{A}$,

$$\langle J'(u), v - u \rangle \geq 0. \quad (3.7)$$

Since the above inequality is hold for any $v \in \mathcal{A}$, we can replace v in the above inequality by $2u - v$ and get

$$\langle J'(u), v - u \rangle \leq 0. \quad (3.8)$$

Thanks to (3.7) and (3.8), we obtain (3.6) \square

Corollary 3.6. *Under Assumptions 3.1-3.2, a necessary and sufficient conditions for an admissible control $u \in \mathcal{A}$ to be an optimal control of Problem 3.1 is the Fréchet derivative of J at the admissible control $u \in \mathcal{A}$ given by*

$$J'(u) = 0. \quad (3.9)$$

Proof. In the equality (3.6), replacing v by $v + u$, we have $\langle J'(u), v \rangle = 0, \forall v \in \mathcal{A}$, i.e. $J'(u) = 0$. Thus the equality (3.6) and the equality (3.9) is equivalent. So the proof can be completed directly by Theorem 3.5. \square

4 Stochastic Hamilton Systems

This section will focus on establishing the dual characterization of the optimal control by stochastic Hamilton system.

Let (u, X) be an admissible pair, then the corresponding adjoint BSDE of the stochastic systems (3.1) is defined by

$$\begin{cases} dp_t &= - \left[A_t^* p_t + \sum_{i=1}^d C_t^{i*} q_t^i + \int_Z E_t^*(\theta) r_t(\theta) \nu(d\theta) + 2Q_t X_t \right] dt \\ &+ \sum_{i=1}^d q_t^i dW_t^i + \int_Z r_t(\theta) \tilde{\mu}(d\theta, dt), \\ p_T &= 2MX_T, \end{cases} \quad (4.1)$$

Note that under Assumption 3.1, from Lemma 2.2, we see that the equation (4.1) admits a unique solution

$$(p, q, r) \in S_{\mathcal{F}}^2(0, T; R^n) \times \mathcal{L}_{\mathcal{F}}^2(0, T; R^n) \times \mathcal{L}_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; R^n).$$

We define the Hamiltonian function $H : [0, T] \times R^n \times R^m \times R^n \times R^{n \times d} \times \mathcal{L}^{\nu, 2}(Z; R^n) \rightarrow R$ by

$$\begin{aligned} &H(t, x, u, p, q, r) \\ &= \langle p, A_t x + B_t u \rangle + \sum_{i=1}^d \langle q^i, C_t^i x + D_t^i u \rangle + \int_Z \langle r(\theta), E_t(\theta) x + F_t(\theta) u \rangle \nu(d\theta) \\ &\quad + \langle Q_t x, x \rangle + \langle N_t u, u \rangle. \end{aligned} \quad (4.2)$$

Then we can rewrite the adjoint equation (4.1) in Hamiltonian system's form:

$$\begin{cases} dp_t &= -H_x(t, X_t, u_t, p_t, q_t, r_t) dt + \sum_{i=1}^d q_t^i dW_t^i + \int_Z r_t(\theta) \tilde{\mu}(d\theta, dt), \\ p_T &= 2MX_T. \end{cases} \quad (4.3)$$

Now we give the dual characterization of the optimal control.

Theorem 4.1. *Let Assumptions 3.1-3.2 be satisfied. Then, a necessary and sufficient condition for an admissible pair (u, X) to be an optimal pair of Problem 3.1 is*

$$H_u(t, X_{t-}, u_t, p_{t-}, q_t, r_t) = 0, \quad a.e.a.s., \quad (4.4)$$

i.e.,

$$2N_t u_t + B_t^* p_{t-} + \sum_{i=1}^d D_t^{i*} q_t^i + \int_Z F_t^*(\theta) r_t(\theta) \nu(d\theta) = 0, \quad a.e.a.s.. \quad (4.5)$$

Here (p, q, r) is the solution of the adjoint equation (4.1) corresponding to the admissible pair (u, X) .

Proof. By Corollary 3.6, in order to prove Theorem 4.1, we only need to show the equality (3.9) and the equality (4.4) or (4.5) are equivalent. Indeed, let (u, X) is an admissible pair. From lemma 3.3, for any admissible control $v \in \mathcal{A}$, we have

$$\langle J'(u), v \rangle = 2E \int_0^T \left[\langle N_t X_t, X_t^{(0,v)} \rangle + \langle Q_t u_t, v_t \rangle \right] dt + 2E \langle MX_T, X_T^{(0,v)} \rangle. \quad (4.6)$$

On the other hand, recalling the adjoint equation (4.1) and the state equation (3.1), applying Itô's formula to $\langle X_t^{(0,v)}, p_t \rangle$ and taking expectation, we have

$$\begin{aligned}
& 2E\langle MX_T, X_T^{(0,v)} \rangle = E\langle p_T, X_T^{(0,v)} \rangle \\
& = E \int_0^T \langle p_s, A_s X_s^{(0,v)} + B_s v_s \rangle ds + \sum_{i=1}^d E \int_0^T \langle q_s^i, C_s^i X_s^{(0,v)} + D_s^i v_s \rangle ds \\
& \quad + E \int_0^T \langle X_s^{(0,v)}, -A_s^* p_s - \sum_{i=1}^d C_s^{i*} q_s^i - \int_Z E_s^*(\theta) r(\theta) \nu(d\theta) - 2Q_s X_s \rangle ds \\
& \quad + E \int \int_{Z \times (0,T]} \langle r_s(\theta), E_s(\theta) X_s^{(0,v)} + F_s(\theta) v_s \rangle \nu(d\theta) ds \\
& = E \int_0^T \langle B_s^* p_s + \sum_{i=1}^d D_s^{i*} q_s^i + \int_Z F_s^*(\theta) r_s(\theta) \nu(d\theta), v_s \rangle ds \\
& \quad - 2E \int_0^T \langle Q_s X_s, X_s^{(0,v)} \rangle ds,
\end{aligned} \tag{4.7}$$

Hence

$$\begin{aligned}
& 2E\langle MX_T, X_T^{(0,v)} \rangle + 2E \int_0^T \langle Q_t X_s, X_s^{(0,v)} \rangle ds + 2E \int_0^T \langle N_s u_s, v_s \rangle ds \\
& = E \int_0^T \langle B_s^* p_s + \sum_{i=1}^d D_s^{i*} q_s^i + \int_Z F_s^*(\theta) r_s(\theta) \nu(d\theta) + 2N_s u_s, v_s \rangle ds.
\end{aligned} \tag{4.8}$$

Combining (4.6) and (4.8), we get

$$\begin{aligned}
& \langle J'(u), v \rangle \\
& = E \int_0^T \langle B_s^* p_s + \sum_{i=1}^d D_s^{i*} q_s^i + \int_Z F_s^*(\theta) r_s(\theta) \nu(d\theta) + 2N_s u_s, v_s \rangle ds \\
& = E \int_0^T \langle H_u(s, X_{s-}, u_s, p_{s-}, q_s, r_s), v_s \rangle ds, \quad \forall v \in \mathcal{A}.
\end{aligned} \tag{4.9}$$

Since the $v \in \mathcal{A}$ in (4.9) is arbitrary, we deduce that the equality (3.9) and the equality (4.5) or (4.4) are equivalent. Then the desired result then follows. \square

Corollary 4.2. *Let assumptions 3.1-3.2 be satisfied. Then, Problem 3.1 has a unique optimal control pair (u, X) , where the optimal control u have the dual representation*

$$u_t = -\frac{1}{2} N_t^{-1} \left[B_t^* p_{t-} + \sum_{i=1}^d D_t^{i*} q_t^i + \int_Z F_t^*(\theta) r_t(\theta) \nu(d\theta) \right], 0 \leq t \leq T. \tag{4.10}$$

Here (p, q, r) is the unique solution of the adjoint equation (4.1) corresponding to the optimal control pair (u, X) .

Proof. From Theorem 3.4, we know that Problem 3.1 have an unique optimal control pair (u, X) . Furthermore, by Theorem 4.1 and the equality (4.5), the optimal control is given by (4.10). \square

Now we can introduce the following so-called stochastic Hamilton system which consists of the state equation (3.1), the dual equation (4.1) and the dual representation (4.10) by

$$\left\{ \begin{array}{l} dX_t = (A_t X_t + B_t u_t)dt + \sum_{i=1}^d (C_t^i X_t + D_t^i u_t) dW_t^i \\ \quad + \int_Z (E_t(\theta) X_{t-} + F_t(\theta) u_t) \tilde{\mu}(d\theta, dt), \\ u_t = -\frac{1}{2} N_t^{-1} \left[B_t^* p_{t-} + \sum_{i=1}^d D_t^{i*} q_t^i + \int_Z F_t^*(\theta) r_t(\theta) \nu(d\theta) \right], \\ dp_t = - \left[A_t^* p_t + \sum_{i=1}^d C_t^{i*} q_t^i + \int_Z E_t^*(\theta) r_t(\theta) \nu(d\theta) + 2Q_t X_t \right] dt, \\ \quad + \sum_{i=1}^d q_t^i dW_t^i + \int_Z r_t(\theta) \tilde{\mu}(d\theta, dt), \\ X_0 = x, \quad p_T = 2M X_T. \end{array} \right. \quad (4.11)$$

Clearly it is a fully coupled forward-backward stochastic differential equations (FB-SDEs in short form) driven by Brownian motion W and Poisson random martingale measure $\tilde{\mu}(d\theta, dt)$. The solutions consist of the stochastic process quaternary (X, p, q, r) .

Theorem 4.3. *Let assumptions 3.1-3.2 be satisfied. Then the stochastic Hamilton system (4.11) has a unique solution $(X, p, q, r) \in S_{\mathcal{F}}^2(0, T; R^n) \times S_{\mathcal{F}}^2(0, T; R^n) \times \mathcal{L}_{\mathcal{F}}^2(0, T; R^{n \times d}) \times \mathcal{L}_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; R^n)$. And u in (4.11) is the optimal control of the stochastic LQ Problem 3.1, the stochastic process X is the corresponding optimal state. Moreover, the following a priori estimate holds*

$$E \sup_{0 \leq t \leq T} |X_t|^2 + E \sup_{0 \leq t \leq T} |p_t|^2 + E \int_0^T |q_t|^2 dt + E \iint_{Z \times (0, T]} |r_t(\theta)|^2 \nu(d\theta) dt \leq K |x|^2, \quad (4.12)$$

where K is some deterministic positive constant.

Proof. The existence result can be directly obtained by Corollary (4.2). The uniqueness result is obvious once the a priori estimate (4.12) holds. Therefore, it remains to prove that the a priori estimate (4.12) hold.

Let (X, p, q, r) is a solution of the stochastic Hamilton systems (4.11). Using Itô's formula $\langle p_t, X_t \rangle$, we get

$$2E \langle M X_T, X_T \rangle + 2E \int_0^T \langle N_t u_t, u_t \rangle + 2E \int_0^T \langle Q_t X_t, X_t \rangle dt = E \langle p_0, x \rangle. \quad (4.13)$$

In the following, K will denote a generic positive constant and might change from line to line.

For the backward part of the stochastic Hamilton systems (4.11), using the a priori estimate (2.7) for BSDEs, we have

$$\begin{aligned}
& E \sup_{0 \leq t \leq T} |p_t|^2 + E \int_0^T |q_t|^2 dt + E \iint_{Z \times (0, T]} |r_t(\theta)|^2 \nu(d\theta) dt \\
& \leq K \left[E \int_0^T |Q_t X_t|^2 dt + E |M X_T|^2 \right] \\
& \leq K \left[E \int_0^T \langle Q_t X_t, X_t \rangle dt + E \langle M X_T, X_T \rangle \right] \\
& \leq K E \langle p_0, x \rangle \\
& \leq K E |p_0| |x| \\
& \leq \frac{1}{2} E |p_0|^2 + K |x|^2 \\
& \leq \frac{1}{2} E \sup_{0 \leq t \leq T} |p_t|^2 + K |x|^2,
\end{aligned} \tag{4.14}$$

where we have used the nonnegative property of Q and M , the equality (4.13) and the elementary inequality

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \quad \forall \varepsilon > 0, a > 0, b > 0.$$

Hence we get

$$E \sup_{0 \leq t \leq T} |p_t|^2 + E \int_0^T |q_t|^2 dt + E \iint_{(0, T] \times Z} |r_t(\theta)|^2 \nu(d\theta) dt \leq K |x|^2. \tag{4.15}$$

On the other hand, for the forward part of the stochastic Hamilton systems (4.11), using the a priori estimate (2.3) for SDEs, we have

$$\begin{aligned}
E \sup_{0 \leq t \leq T} |x_t|^2 & \leq K \left[E \int_0^T |u_t|^2 dt + |x|^2 \right] \\
& \leq K \left[E \int_0^T \langle N_t u_t, u_t \rangle dt + |x|^2 \right] \\
& \leq K \left[E \langle p_0, x \rangle + |x|^2 \right] \\
& \leq K \left[E |p_0| |x| + |x|^2 \right] \\
& \leq K \left[E |p_0|^2 + |x|^2 \right] \\
& \leq K \left[E \sup_{0 \leq t \leq T} |p_t|^2 + |x|^2 \right] \\
& \leq K |x|^2,
\end{aligned} \tag{4.16}$$

where we have used the nonnegative property of N , the equality (4.13), the elementary inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$, $\forall \varepsilon > 0, a > 0, b > 0$, and the inequality (4.15).

Combining the inequality (4.15) and the inequality (4.16), the inequality (4.12) is directly obtained. The proof is complete. \square

In summary, the stochastic Hamilton system (4.11) completely characterizes the optimal control of LQ problem. Therefore, solving LQ problem is equivalent to solving the stochastic Hamilton system, moreover, the unique optimal control can be given explicitly by (4.10).

5 Backward Stochastic Riccati equation with jumps

Although the stochastic Hamilton system (4.11) is a complete characterization of the stochastic LQ problem, it is a fully coupled forward-backward stochastic differential equation. The solution to (4.11) would be hard to be solved so that this characterization is also not satisfactory. As the stochastic LQ theory in Brownian motion framework (see [18]), it is natural to connect the stochastic LQ problem with stochastic Riccati equation. In this section, we will introduce stochastic Riccati equation with jumps and establish the link with the stochastic Hamilton system (4.11), then show the optimal control of the stochastic LQ problem has state feedback representation. In the end, we will focus on discussing the existence and uniqueness of the solution to the stochastic Riccati equation with jumps.

5.1 Derivation of stochastic Riccati equation with jumps

In the following, by dynamic programming principle, we will derive the general form of the stochastic Riccati equation with jumps.

Now consider the following parameterized stochastic LQ problem on the initial time t and the initial state x :

The state equation

$$\begin{cases} dX_s = (A_s X_s + B_s u_s) ds + \sum_{i=1}^d (C_s^i X_s + D_s^i u_s) dW_s^i \\ \quad + \int_E (E_s(\theta) X_{s-} + F_s(\theta) u_s) \tilde{\mu}(d\theta, ds), \\ X_t = x, \quad 0 \leq s \leq T. \end{cases} \quad (5.1)$$

The cost functional

$$J(t, x; u) := E^{\mathcal{F}_t} \int_t^T \left[\langle Q_s X_s, X_s \rangle + \langle N_s u_s, u_s \rangle \right] ds + E^{\mathcal{F}_t} \langle M X_T, X_T \rangle. \quad (5.2)$$

Define the value function by

$$\Phi_t(x) := \inf_{u \in \mathcal{A}} J(t, x; u). \quad (5.3)$$

Then the value function $\{\Phi_t(x), t \in [0, T], x \in R^n\}$ is a family of $\{\mathcal{F}_t, 0 \leq t \leq T\}$ -adapted processes with values in R . In general, for any $x \in R^n$, $\Phi_t(x)$ is not a bounded

variation function with respect to t . So we can only expect that $\{\Phi_t(x), t \in [0, T], x \in R^n\}$ is a family of semimartingales with the decomposition

$$\Phi_t(x) = \langle Mx, x \rangle + \int_t^T \Gamma_s(x) ds - \sum_{i=1}^d \int_t^T \Lambda_s^i(x) dW_s^i - \iint_{(t,T] \times Z} \Psi_s(\theta, x) \tilde{\mu}(d\theta, ds). \quad (5.4)$$

Furthermore, suppose

$$\begin{aligned} \Phi_t(x) &= \langle K_t x, x \rangle; \\ \Lambda_t^i(x) &= \langle L_t^i x, x \rangle, i = 1, 2, \dots, d; \\ \Psi_s(\theta, x) &= \langle H_t(\theta) x, x \rangle, \quad t \in [0, T], x \in R^n, \theta \in Z, \end{aligned} \quad (5.5)$$

where K is a symmetric matrix-valued $\{\mathcal{F}_t, 0 \leq t \leq T\}$ -adapted process, $L^i (i = 1, 2, \dots, d)$ and H are symmetric matrix-valued $\{\mathcal{F}_t, 0 \leq t \leq T\}$ -predictable processes. Firstly, using the dynamic programming principle (see[15]) and Itô-Ventzell formulation with jumps (see[5]), we deduce that $\Gamma_t(x)$ in the semimartingale decomposition (5.4) have the following expression

$$\begin{aligned} \Gamma_t(x) &= \inf_{u \in R^m} \left\{ \langle D\Phi_t(x), A_t x + B_t u \rangle + \frac{1}{2} \sum_{i=1}^d \langle D^2\Phi_t(x), (C_t^i x + D_t^i u)(C_t^i x + D_t^i u)^* \rangle \right. \\ &\quad + \sum_{i=1}^d \langle D\Lambda_t^i(x), C_t^i x + D_t^i u \rangle + \langle Q_t x, x \rangle + \langle N_t u, u \rangle \\ &\quad + \int_Z [\Phi_t(x + E_t(\theta)x + F_t(\theta)u) - \Phi_t(x) - \langle D\Phi_t(x), E_t(\theta)x + F_t(\theta)u \rangle] \nu(d\theta) \\ &\quad \left. + \int_Z [\Psi_t(\theta, x + E_t(\theta)x + F_t(\theta)u) - \Psi_t(\theta, x)] \nu(d\theta) \right\}, \end{aligned} \quad (5.6)$$

where $D\Phi_t(x)$ and $D\Lambda_t(x)$ is the gradient of $\Phi_t(x)$ and $\Lambda_t(x)$ with respect to x respectively, $D^2\Phi_t(x)$ is the Hessian of $\Phi_t(x)$ with respect to x . Now substituting the relationship (5.5) into (5.6), we get

$$\begin{aligned} \Gamma_t(x) &= \inf_{u \in R^m} \left\{ \left\langle x, \left[K_t A_t + A_t^* K_t + \sum_{i=1}^d L_t^i C_t^i + \sum_{i=1}^d C_t^{i*} L_t^i + \sum_{i=1}^d C_t^{i*} K_t C_t^i \right. \right. \right. \\ &\quad + \int_Z H_t(\theta) E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H_t(\theta) \nu(d\theta) \\ &\quad + \int_Z E_t^*(\theta) K_t E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H_t(\theta) E_t(\theta) \nu(d\theta) + Q_t \left. \right] x \rangle \\ &\quad + 2 \left\langle u, \left[B_t^* K_t + \sum_{i=1}^d D_t^{i*} L_t^i + \sum_{i=1}^d D_t^{i*} K_t C_t^i \right. \right. \\ &\quad + \int_Z F_t^*(\theta) H_t(\theta) \nu(d\theta) + \int_Z F_t^*(\theta) K_t E_t(\theta) \nu(d\theta) \\ &\quad + \int_Z F_t^*(\theta) H_t(\theta) E_t(\theta) \nu(d\theta) \left. \right] x \rangle + \left\langle u, \left[N_t + \sum_{i=1}^d D_t^{i*} K_t D_t^i \right. \right. \\ &\quad \left. \left. + \int_Z F_t^*(\theta) K_t F_t(\theta) \nu(d\theta) + \int_Z F_t^*(\theta) H_t(\theta) F_t(\theta) \nu(d\theta) \right] u \right\rangle \right\}. \end{aligned} \quad (5.7)$$

It is obvious that for $\forall(t, x, \omega) \in [0, T] \times R^n \times \Omega$, $\Gamma_t(x)$ is the Quadratic functional extreme with respect to $u \in R^m$.

Furthermore, if $N_t + \sum_{i=1}^d D_t^{i*} K_t D_t^i + \int_Z F_t^*(\theta) K_t F_t(\theta) \nu(d\theta) + \int_Z F_t^*(\theta) H_t(\theta) F_t(\theta) \nu(d\theta)$ is strictly positive definite, then it follows that the infimum in (5.7) is obtained at

$$\begin{aligned} u = & - \left[N_t + \sum_{i=1}^d D_t^{i*} K_t D_t^i + \int_Z F_t^*(\theta) K_t F_t(\theta) \nu(d\theta) \right. \\ & \left. + \int_Z F_t^*(\theta) H_t(\theta) F_t(\theta) \nu(d\theta) \right]^{-1} \left(B_t^* K_t + \sum_{i=1}^d D_t^{i*} K_t C_t^i + \sum_{i=1}^d D_t^{i*} L_t^i \right. \\ & \left. + \int_Z F_t^*(\theta) K_t E_t(\theta) \nu(d\theta) + \int_Z F_t^*(\theta) H_t(\theta) \nu(d\theta) \right. \\ & \left. + \int_Z F_t^*(\theta) H_t(\theta) E_t(\theta) \tilde{\nu}(d\theta) \right) x \end{aligned} \quad (5.8)$$

Combining (5.4), (5.5), (5.7) and (5.8), we deduce that the matrix-valued processes (K, L, H) satisfy the following Riccati equation

$$\begin{cases} dK_t = -G_t - Q_t + \hat{B}_t \hat{N}_t^{-1} \hat{B}_t^* dt + \sum_{i=1}^d L_t^i dW_t^i + \int_Z H_t(\theta) \mu(d\theta, dt), \\ K_T = M, \end{cases} \quad (5.9)$$

where

$$\begin{aligned} G_t := & K_t A_t + A_t^* K_t + \sum_{i=1}^d L_t^i C_t^i + \sum_{i=1}^d C_t^{i*} L_t^i + \sum_{i=1}^d C_t^{i*} K_t C_t^i \\ & + \int_Z H_t(\theta) E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H_t(\theta) \nu(d\theta) \\ & + \int_Z E_t^*(\theta) K_t E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H_t(\theta) E_t(\theta) \nu(d\theta), \end{aligned} \quad (5.10)$$

$$\begin{aligned} \hat{B}_t = & K_t B_t + \sum_{i=1}^d L_t^i D_t^i + \sum_{i=1}^d C_t^{i*} K_t D_t^i \\ & + \int_Z H_t(\theta) F_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) K_t F_t(\theta) \nu(d\theta) \\ & + \int_Z E_t^*(\theta) H_t(\theta) F_t(\theta) \nu(d\theta), \end{aligned} \quad (5.11)$$

$$\begin{aligned} \hat{N}_t = & N_t + \sum_{i=1}^d D_t^{i*} K_t D_t^i + \int_Z F_t^*(\theta) K_t F_t(\theta) \nu(d\theta) \\ & + \int_Z F_t^*(\theta) H_t(\theta) F_t(\theta) \nu(d\theta). \end{aligned} \quad (5.12)$$

It is a high order nonlinear backward stochastic differential equations with the generator $-G_t - Q_t + \hat{B}_t \hat{N}_t^{-1} \hat{B}_t^*$, the unknown elements are the triple matrix process (K, L, H) . The above backward stochastic Riccati differential equation with jumps will be hereafter abbreviated as BSRDEJ.

Now we give the rigorous connection of BSRDEJ (5.9) to the stochastic Hamilton system (4.11) and to the stochastic LQ Problem 3.1.

Theorem 5.1. *Let Assumptions 3.1–3.2 be satisfied. Let (X, p, q, r) be the solution of the stochastic Hamilton system (4.11) with u being the optimal control. Assume that $(K, L, H) \in \mathcal{S}_{\mathcal{F}}^2(0, T; \mathcal{S}^n) \times \mathcal{L}_{\mathcal{F}}^2(0, T; (\mathcal{S}^n)^d) \times \mathcal{L}_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; \mathcal{S}^n)$ is the solution to BSRDEJ (5.9) and the matrix-valued process \hat{N} (noting (5.12)) is a.e.a.s. positive definite. Then, we have, for $t \in [0, T]$ and $\theta \in Z$*

$$\begin{aligned} p_t &= K_t X_t; \\ q_t^i &= (L_t^i + K_{t-} C_t^i) X_{t-} + K_{t-} D_t^i u_t, \quad i = 1, 2, \dots, d; \\ r_t(\theta) &= \left(H_t(\theta) + K_{t-} E_t(\theta) + H_t(\theta) E_t(\theta) \right) X_{t-} \\ &\quad + \left(K_{t-} F_t(\theta) + H_t(\theta) F_t(\theta) \right) u_t. \end{aligned} \tag{5.13}$$

Proof. Use Itô formula to compute $K_t x_t$ and compare it with p_t . The identification of the integrands of Lebesgue and Itô's integrals yields the desired relation (5.13). \square

Now we give the state feedback representation of optimal control u .

Theorem 5.2. *Let Assumptions 3.1–3.2 hold. Let (u, X) be the optimal pair of the stochastic LQ Problem 3.1. Assume that $(K, L, H) \in \mathcal{S}_{\mathcal{F}}^2(0, T; \mathcal{S}^n) \times \mathcal{L}_{\mathcal{F}}^2(0, T; (\mathcal{S}^n)^d) \times \mathcal{L}_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; \mathcal{S}^n)$ is the solution to BSRDEJ (5.9) and the matrix-valued process \hat{N} (noting (5.12)) is a.e.a.s. positive definite. Then u has the following state feedback representation*

$$\begin{aligned} u_t &= \left[N_t + \sum_{i=1}^d D_t^{i*} K_{t-} D_t^i + \int_Z F_t^*(\theta) K_{t-} F_t(\theta) \nu(d\theta) \right. \\ &\quad \left. + \int_Z F_t^*(\theta) H_t(\theta) F_t(\theta) \nu(d\theta) \right]^{-1} \left[B_t^* K_{t-} + \sum_{i=1}^d D_t^{i*} K_{t-} C_t^i \right. \\ &\quad \left. + \sum_{i=1}^d D_t^{i*} L_t^i + \int_Z F_t^*(\theta) K_{t-}(\theta) E_t(\theta) \nu(d\theta) + \int_Z F_t^*(\theta) H_t(\theta) \nu(d\theta) \right. \\ &\quad \left. + \int_Z F_t^*(\theta) H_t(\theta) E_t(\theta) \nu(d\theta) \right] X_{t-}, \quad a.e.a.s.. \end{aligned} \tag{5.14}$$

Moreover, the following relation holds

$$\inf_{u \in \mathcal{U}} J(u) = E \langle K_0 x, x \rangle.$$

Proof. Putting into the relationship (5.13) into the dual representation (4.5), we get the state feedback representation (5.14). Since (u, X) is the optimal pair, combining the relationship (4.13) and the first relationship in (5.13), we get

$$\begin{aligned} \inf_{u \in \mathcal{U}} J(u) &= 2E\langle Mx_T, x_T \rangle + 2E \int_0^T \langle N_t u_t, u_t \rangle dt + 2E \int_0^T \langle Q_t x_t, x_t \rangle dt \\ &= E\langle P_0, x \rangle = E\langle K_0 x, x \rangle. \end{aligned} \quad (5.15)$$

The proof is complete.

Remark 5.1. Formula (5.14) provides a characterization of the optimal control in the terms of the solution to BSRDEJ (5.9). BSRDEJ (5.9) is not a coupled equation, and this characterization is preferred to (4.11).

Remark 5.2. Putting (5.14) into the second equality and the third equality of (5.13), we have

$$\begin{aligned} q_t^i &= (L_t^i + K_{t-} C_t^i) X_{t-} + K_{t-} D_t^i \left[N_t + \sum_{i=1}^d D_t^{i*} K_{t-} D_t^i + \int_Z F_t^*(\theta) K_{t-} F_t(\theta) \nu(d\theta) \right. \\ &\quad \left. + \int_Z F_t^*(\theta) H_t(\theta) F_t(\theta) \nu(d\theta) \right]^{-1} \left[B_t^* K_{t-} + \sum_{i=1}^d D_t^{i*} K_{t-} C_t^i \right. \\ &\quad \left. + \sum_{i=1}^d D_t^{i*} L_t^i + \int_Z F_t^*(\theta) K_{t-}(\theta) E_t(\theta) \nu(d\theta) + \int_Z F_t^*(\theta) H_t(\theta) \nu(d\theta) \right. \\ &\quad \left. + \int_Z F_t^*(\theta) H_t(\theta) E_t(\theta) \nu(d\theta) \right] X_{t-}, \quad i = 1, 2, \dots, d; \\ r_t(\theta) &= \left(H_t(\theta) + K_{t-} E_t(\theta) + H_t(\theta) E_t(\theta) \right) X_{t-} \\ &\quad + \left(K_{t-} F_t(\theta) + H_t(\theta) F_t(\theta) \right) \left[N_t + \sum_{i=1}^d D_t^{i*} K_{t-} D_t^i + \int_Z F_t^*(\theta) K_{t-} F_t(\theta) \nu(d\theta) \right. \\ &\quad \left. + \int_Z F_t^*(\theta) H_t(\theta) F_t(\theta) \nu(d\theta) \right]^{-1} \left[B_t^* K_{t-} + \sum_{i=1}^d D_t^{i*} K_{t-} C_t^i + \sum_{i=1}^d D_t^{i*} L_t^i \right. \\ &\quad \left. + \int_Z F_t^*(\theta) K_{t-}(\theta) E_t(\theta) \nu(d\theta) + \int_Z F_t^*(\theta) H_t(\theta) \nu(d\theta) \right. \\ &\quad \left. + \int_Z F_t^*(\theta) H_t(\theta) E_t(\theta) \nu(d\theta) \right] X_{t-}, \quad t \in [0, T], \quad \theta \in Z. \end{aligned}$$

5.2 Existence and uniqueness of BSRDE with jump

From Theorem 5.2, we know that the optimal control u of the stochastic LQ Problem 3.1 can be expressed by the solution (K, L, H) to the BSRDEJ (5.9). Therefore, solving stochastic LQ Problem 3.1 is equivalent to solving the BSRDEJ (5.9). But the BSRDEJ (5.9) is a high order nonlinear backward stochastic differential equation with jumps. And the general theory of BSDE (see lemma 2.2) can be not applied to use to guarantee the existence and uniqueness of its solution. Moreover, different from the BSRDE driven only by Brownian motion (see Tang [18]), the BSRDEJ (5.9) has also a notable characteristic: the nonlinear term $\hat{N}_t^{-1} = (N_t + D_t^{i*} K_{t-} D_t^i + \int_Z F_t^*(\theta) K_{t-} F_t(\theta) \nu(d\theta) +$

$\int_Z F_t(\theta)H_t(\theta)F_t(\theta)\nu(d\theta))^{-1}$ contains not only the first unknown element K , but also the third unknown element H . For the BSRDE driven by only Brownian motion, the nonlinear term \hat{N}_t^{-1} is degenerated into $(N_t + D_t^{i*}K_tD_t^i)^{-1}$ which only contain the first unknown element K_t . In [18], we can proof the K_t is non-negative matrix, so $(N_t + D_t^{i*}K_tD_t^i)^{-1}$ is well defined. For the second unknown element L , we can only show it's square integrability, but we can not show if it is a non-negative matrix. So for the BSRDEJ (5.9), how to guarantee $(N_t + D_t^{i*}K_tD_t^i + \int_Z F_t^*(\theta)K_tF_t(\theta)\nu(d\theta) + \int_Z F_t^*(\theta)H_t(\theta)F_t(\theta)\nu(d\theta))^{-1}$ to be well-defined is posed to be a challenging problem.

In this paper, we show the existence and uniqueness result only for the case where the generator is a bounded linear dependence with respect to the second unknown element L and the third unknown element H . For the general case, we will to continue the discussion in future research .

Now we give the further assumptions on the coefficients of stochastic system (3.1). Assume that the coefficients

$$\begin{aligned} C &= (C^1, \dots, C^d) = (C^{11}, \dots, C^{1d_1}, C^{21}, \dots, C^{2d_2}), \\ D &= (D^1, \dots, D^d) = (D^{11}, \dots, D^{1d_1}, 0, \dots, 0), \\ F &= 0, \end{aligned}$$

where $d_1 + d_2 = d$.

In this case the stochastic system (3.1) is reduced to the following form

$$\begin{cases} dX_t = (A_t X_t + B_t u_t)dt + \sum_{i=1}^{d_1} C_t^{1i} X_t dW_t^{1i} + \sum_{i=1}^{d_2} (C_t^{2i} X_t + D_t^{2i} u_t) dW_t^{2i} \\ \quad + \int_Z E_t(\theta) X_{t-} \tilde{\mu}(d\theta, dt) \\ x_0 = x, \end{cases} \quad (5.16)$$

Denote by $\{\mathcal{F}_t^*\}_{t \geq 0}$ the P-augmentation of the natural σ -filtration which is generated by Brownian motion $(W^{11}, \dots, W^{1d_1})$ and Poisson random martingale measure $\tilde{\mu}(d\theta, dt)$. In the following we give the further assumptions on adaption of the coefficients of the stochastic LQ problem.

Assumption 5.1. Assume that A, B, C, D, E, Q, N are uniforming bounded $\{\mathcal{F}_t^*, 0 \leq t \leq T\}$ -predictable matrix-valued processes. And the random matrix M is bounded \mathcal{F}_T^* -measurable.

Under Assumption 5.1, again by dynamic programming principle and Itô-Ventzell

formulation, BSRDEJ (5.9) is reduced to the following form

$$\begin{aligned}
dK_t = & - \left[K_t A_t + A_t^* K_t + \sum_{i=1}^{d_1} C_t^{1i*} L_t^{1i} + \sum_{i=1}^{d_1} L_t^{1i} C_t^{1i} + \sum_{i=1}^{d_1} C_t^{1i*} K_t C_t^{1i} \right. \\
& + \sum_{i=1}^{d_2} C_t^{2i*} K_t C_t^{2i} + \int_Z H_t(\theta) E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H_t(\theta) \nu(d\theta) \\
& + \int_Z E_t^*(\theta) K_t E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H_t(\theta) E_t(\theta) \nu(d\theta) \\
& \left. + Q_t - \hat{B}(t, K_t) \hat{N}^{-1}(t, K_t) \hat{B}^*(t, K_t) \right] dt \\
& + \sum_{i=1}^{d_1} L_t^i dW_t^{1i} + \int_Z H_t(\theta) \tilde{\mu}(d\theta, dt),
\end{aligned} \tag{5.17}$$

where for $\forall K \in \mathcal{S}^n$, we define

$$\begin{aligned}
\hat{B}(t, K) &:= K B_t + \sum_{i=1}^{d_2} C_t^{2i} K D^{2i*}, \\
\hat{N}(t, K) &:= N_t + \sum_{i=1}^{d_2} D^{2i*} K D^{2i}.
\end{aligned} \tag{5.18}$$

In the following we state the existence and uniqueness result of the solution BSRDEJ (5.17).

Theorem 5.3. *Let Assumption 3.2 and Assumption 5.1 hold. Then BSRDEJ (5.17) has a unique solution $(K, L, H) \in \mathcal{S}_{\mathcal{F}}^2(0, T; \mathcal{S}^n) \times \mathcal{L}_{\mathcal{F}}^2(0, T; (\mathcal{S}^n)^{d_1}) \times \mathcal{L}_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; \mathcal{S}^n)$. Moreover, K is uniformly bounded and nonnegative a.s.a.e..*

In order to show the theorem, we need the following two lemmas.
Consider the following linear BSDE

$$\left\{ \begin{aligned} -d\hat{K}_t = & \left[\hat{K}_t \hat{A}_t + \hat{A}_t^* \hat{K}_t + \sum_{i=1}^{d_1} \hat{L}_t^{1i} \hat{C}_t^{1i} + \sum_{i=1}^{d_1} \hat{C}_t^{1i*} \hat{L}_t^{1i} + \sum_{i=1}^{d_1} \hat{C}_t^{1i*} \hat{K}_t \hat{C}_t^{1i} \right. \\
& + \sum_{i=1}^{d_2} \hat{C}_t^{2i*} \hat{K}_t \hat{C}_t^{2i} + \int_Z \hat{H}_t(\theta) \hat{E}_t(\theta) \nu(d\theta) + \int_Z \hat{E}_t^*(\theta) \hat{H}_t(\theta) \nu(d\theta) \\
& + \int_Z \hat{E}_t^*(\theta) \hat{K}_t \hat{E}_t(\theta) \nu(d\theta) + \int_Z \hat{E}_t^*(\theta) \hat{H}_t(\theta) \hat{E}_t(\theta) \nu(d\theta) + \hat{Q}_t \Big] dt \\
& - \sum_{i=1}^{d_1} \hat{L}_t^{1i} dW_t^{1i} - \int_Z \hat{H}_t(\theta) \tilde{\mu}(d\theta, dt), \\ \hat{K}_T = & \hat{M}. \end{aligned} \right. \tag{5.19}$$

Lemma 5.4. *Let $\hat{A}, \hat{C}^{1i} (i = 1, 2, \dots, d_1), \hat{C}^{2i} (i = 1, 2, \dots, d_2), \hat{E}$ be $R^{n \times n}$ -valued, and \hat{Q} be \mathcal{S}^n -valued, uniformly bounded $\{\mathcal{F}_t^*, 0 \leq t \leq T\}$ -predictable process. Let \hat{M} be \mathcal{S}^n -valued bounded \mathcal{F}_T^* -measurable random variable. Then BSDE (5.19) has unique solution*

$(\hat{K}, \hat{L}, \hat{H}) \in \mathcal{S}_{\mathcal{F}^*}^2(0, T; \mathcal{S}^n) \times \mathcal{L}_{\mathcal{F}^*}^2(0, T; (\mathcal{S}^n)^{d_1}) \times \mathcal{L}_{\mathcal{F}^*}^{\nu, 2}([0, T] \times Z; \mathcal{S}^n)$. Moreover,

$$\sup_{(t, \omega) \in [0, T] \times \Omega} |\hat{K}_t(\omega)|^2 \leq \kappa_0 < +\infty, \quad (5.20)$$

where κ_0 depends on

$$\sup_{\omega} \left(|\hat{M}(\omega)|^2 + \int_0^T |\hat{Q}_t(\omega)|^2 dt \right). \quad (5.21)$$

If \hat{Q} and \hat{M} are nonnegative a.s.a.e., then \hat{K} is also nonnegative a.s.a.e..

Proof. According to Lemma 2.2, the existence and uniqueness as well as the inequality (5.20) can be obtained directly. It remains to prove the nonnegativity of \hat{K} . For any given $(t, x) \in [0, T] \times R^n$, we introduce the following linear SDE:

$$\begin{aligned} dy_s = & \hat{A}_s y_s ds + \sum_{i=1}^{d_1} \hat{C}_s^{1i} y_s dW_s^{1i} + \sum_{i=1}^{d_2} \hat{C}_s^{2i} y_s dW_s^{2i} \\ & + \int_Z \hat{E}_s(\theta) y_{s-} \tilde{\mu}(d\theta, ds), \quad y_t = x, \quad t \leq s \leq T. \end{aligned} \quad (5.22)$$

From Lemma 2.1, SDE (5.22) has a unique strong solution y . Applying Itô's formula to $\langle \hat{K}_s y_s, y_s \rangle$ we have

$$\begin{aligned} d\langle \hat{K}_s y_s, y_s \rangle = & -\langle \hat{Q}_s y_s, y_s \rangle ds + \sum_{i=1}^{d_1} \langle y_s, (\hat{L}_s^{1i} + \hat{K}_s \hat{C}_s^{1i} + \hat{C}_s^{1i*} \hat{K}_s) y_s \rangle dW_s^{1i} \\ & + \sum_{i=1}^{d_2} \langle y_s, (\hat{K}_s \hat{C}_s^{2i} + \hat{C}_s^{2i*} \hat{K}_s) y_s \rangle dW_s^{2i} \\ & + \int_Z \langle y_{s-}, (\hat{H}_s(\theta) + \hat{K}_{s-} \hat{E}_s(\theta) + \hat{E}_s^*(\theta) \hat{K}_{s-} + \hat{H}_s(\theta) \hat{E}_s(\theta)) y_{s-} \rangle \tilde{\mu}(d\theta, ds) \\ & + \int_Z \langle y_{s-}, \hat{E}_s^*(\theta) (\hat{H}_s(\theta) + \hat{K}_{s-} \hat{E}_s(\theta) + \hat{H}_s(\theta) \hat{E}_s(\theta)) y_{s-} \rangle \tilde{\mu}(d\theta, ds), \end{aligned} \quad (5.23)$$

Thus, taking conditional expectation, we get

$$\langle \hat{K}_t x, x \rangle = E^{\mathcal{F}_t^*} \left[\int_t^T \langle \hat{Q}_s y_s, y_s \rangle ds + \langle \hat{M} y_T, y_T \rangle \right]. \quad (5.24)$$

Since \hat{Q} and \hat{M} are nonnegative a.s.a.e., from (5.24), we conclude that \hat{K} is nonnegative a.s.a.e.. \square

Define the mapping $F : [0, T] \times (\mathcal{S}^n)_+ \times (\mathcal{S}^n)^{d_1} \times \mathcal{L}^{\nu, 2}(Z; \mathcal{S}^n) \times R^{m \times n} \rightarrow \mathcal{S}^n$ by

$$\begin{aligned}
F(t, K, L^1, H, U) = & (A_t - B_t U)^* K + K(A_t - B_t U) + \sum_{i=1}^{d_1} C_t^{1i*} L^{1i} + \sum_{i=1}^{d_1} L^{1i} C_t^{1i} \\
& + \sum_{i=1}^{d_1} C_t^{1i*} K C_t^{1i} + \int_Z H(\theta) E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H(\theta) \nu(d\theta) \\
& + \int_Z E_t^*(\theta) K E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H(\theta) E_t(\theta) \nu(d\theta) \\
& + \sum_{i=1}^{d_2} (C_t^{2i} - D_t^{2i} U)^* K (C_t^{2i} - D_t^{2i} U).
\end{aligned} \tag{5.25}$$

By notation (5.18), define the mapping $\hat{U} : [0, T] \times (\mathcal{S}^n)^+ \rightarrow R^{m \times n}$ by

$$\hat{U}(t, K) = \hat{N}^{-1}(t, K) \hat{B}^*(t, K).$$

Lemma 5.5. *Let $(K, L^1, H) \in (\mathcal{S}^n)_+ \times (\mathcal{S}^n)^{d_1} \times \mathcal{L}^{\nu, 2}(Z; \mathcal{S}^n)$. Then, for $\forall U \in R^{m \times n}$, we have*

$$F(t, K, L^1, H, U) + U^* N_t U \geq F(t, K, L^1, H, \hat{U}(t, K)) + \hat{U}^*(t, K) N_t \hat{U}(t, K), \quad 0 \leq t \leq T. \tag{5.26}$$

Proof. By the definition of $F(t, K, L^1, H, U)$, $\hat{B}(t, K)$, $\hat{N}(t, K)$ and $\hat{U}(t, K)$, it follows that

$$\begin{aligned}
F(t, K, L^1, H, U) + U^* N_t U = & -U^* \hat{B}^*(t, K) - \hat{B}(t, K) U + U^* \hat{N}(t, K) U \\
& + A_t^* K + K A_t + \sum_{i=1}^{d_1} C_t^{1i*} L^{1i} + \sum_{i=1}^{d_1} L^{1i} C_t^{1i} \\
& + \sum_{i=1}^{d_1} C_t^{1i*} K C_t^{1i} + \int_Z H(\theta) E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H(\theta) \nu(d\theta) \\
& + \int_Z E_t^*(\theta) K E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H(\theta) E_t(\theta) \nu(d\theta) \\
& + \sum_{i=1}^{d_2} C_t^{2i*} K C_t^{2i},
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}
F(t, K, L^1, H, \hat{U}(t, K)) + \hat{U}^*(t, K) N_t \hat{U}(t, K) = & -\hat{U}^*(t, K) \hat{N}(t, K) \hat{U}(t, K) \\
& + A_t^* K + K A_t + \sum_{i=1}^{d_1} C_t^{1i*} L^{1i} + \sum_{i=1}^{d_1} L^{1i} C_t^{1i} \\
& + \sum_{i=1}^{d_1} C_t^{1i*} K C_t^{1i} + \int_Z H(\theta) E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H(\theta) \nu(d\theta) \\
& + \int_Z E_t^*(\theta) K E_t(\theta) \nu(d\theta) + \int_Z E_t^*(\theta) H(\theta) E_t(\theta) \nu(d\theta) \\
& + \sum_{i=1}^{d_2} C_t^{2i*} K C_t^{2i}.
\end{aligned} \tag{5.28}$$

Therefore,

$$\begin{aligned}
& F(t, K, L^1, H, U) + U^* N_t U - F(K, L^1, H, \hat{U}(t, K)) - \hat{U}^*(t, K) N_t \hat{U}(t, K) \\
&= -U^* \hat{B}^*(t, K) - \hat{B}(t, K) U + U^* \hat{N}(t, K) U + \hat{U}^*(t, K) \hat{N}(t, K) \hat{U}(t, K) \\
&= -U^* \hat{N}(t, K) \hat{U}(t, K) - \hat{U}^*(t, K) \hat{N}_t U + U^* \hat{N}(t, K) U + \hat{U}^*(t, K) \hat{N}_t \hat{U}(t, K) \\
&= (\hat{U}(t, K) - U)^* \hat{N}(t, K) (\hat{U}(t, K) - U) \geq 0.
\end{aligned} \tag{5.29}$$

The proof is complete. \square

In the following we will use the Bellmans principle of quasi-linearization and a monotone convergence result of symmetric matrices (see [20]) to show Theorem 5.3.

Existence By the definition of $F(t, K, L^1, H, U)$, BSRDEJ (5.17) can be rewritten as the following quasi-linearization BSDE

$$\begin{aligned}
-dK_t &= \left[F(t, K_t, L_t^1, H_t, \hat{U}(t, K_t)) + \hat{U}^*(t, K_t) N_t \hat{U}(t, K_t) + Q_t \right] dt \\
&\quad - \sum_{i=1}^{d_1} L_t^{1i} dW_t^{1i} - \int_Z H_t(\theta) \tilde{\mu}(d\theta, dt), \quad K_T = M.
\end{aligned} \tag{5.30}$$

Making use of Eq.(5.30), we will iteratively construct a sequence $\{(K_j, L_j^1, H_j)\}_{j=1}^\infty$ of approximating solutions of BSRDEJ (5.17). In fact, by Lemma 5.4, we set $(K_0, L_0^1, H_0) = (0, 0, 0)$ and solve iteratively the following linear BSDE:

$$\left\{ \begin{aligned} -dK_{j+1,t} &= \left[F(t, K_{j+1,t}, L_{j+1,t}^1, H_{j+1,t}, \hat{U}(t, K_{j,t})) + \hat{U}^*(t, K_{j,t}) N_t \hat{U}(t, K_{j,t}) + Q_t \right] dt \\ &\quad - \sum_{i=1}^{d_1} L_{j+1,t}^{1i} dW_t^{1i} - \int_Z H_{j+1,t}(\theta) \tilde{\mu}(d\theta, dt), \\ K_{j+1,T} &= M. \end{aligned} \right. \tag{5.31}$$

From Lemma 5.4, it follows that K_j is a.e.a.s. bounded and nonnegative. We also claim that $\{K_{j+1}\}$ is a.e.a.s. a non-increasing sequence. Indeed,

$$\begin{aligned}
& -d(K_{j,t} - K_{j+1,t}) = (-dK_{j,t}) - (-dK_{j+1,t}) \\
& = \left[F(t, K_{j,t}, L_{j,t}^1, H_{j,t}, \hat{U}(t, K_{j-1,t})) + \hat{U}^*(t, K_{j-1,t})N_t\hat{U}(t, K_{j-1,t}) \right. \\
& \quad \left. - F(t, K_{j+1,t}, L_{j+1,t}^1, H_{j+1,t}, \hat{U}(t, K_{j,t})) - \hat{U}^*(t, K_{j,t})N_t\hat{U}(t, K_{j,t}) \right] dt \\
& \quad - \sum_{i=1}^{d_1} (L_{j,t}^{1i} - L_{j+1,t}^{1i})dW_t^{1i} - \int_Z (H_{j,t}(\theta) - H_{j+1,t}(\theta))\tilde{\mu}(d\theta, dt) \\
& = \left[F(t, K_{j,t}, L_{j,t}^1, H_{j,t}, \hat{U}(t, K_{j,t})) - F(t, K_{j+1,t}, L_{j+1,t}^1, H_{j+1,t}, \hat{U}(t, K_{j,t})) \right. \\
& \quad + \left(F(t, K_{j,t}, L_{j,t}^1, H_{j,t}, \hat{U}(t, K_{j-1,t})) + \hat{U}^*(t, K_{j-1,t})N_t\hat{U}(t, K_{j-1,t}) \right. \\
& \quad \left. - F(t, K_{j,t}, L_{j,t}, H_{j,t}, \hat{U}(t, K_{j,t})) - \hat{U}^*(t, K_{j,t})N_t\hat{U}(t, K_{j,t}) \right) \left. \right] dt \\
& \quad - \sum_{i=1}^{d_1} (L_{j,t}^{1i} - L_{j+1,t}^{1i})dW_t^{1i} - \int_Z (H_{j,t}(\theta) - H_{j+1,t}(\theta))\tilde{\mu}(d\theta, dt) \\
& = \left[F(t, K_{j,t} - K_{j+1,t}, L_{j,t} - L_{j+1,t}, H_{j,t} - H_{j+1,t}, \hat{U}(t, K_{j,t})) \right. \\
& \quad + (\hat{U}(t, K_{j,t}) - \hat{U}(t, K_{j-1,t}))^* \hat{N}(t, K_{j,t})(\hat{U}(t, K_{j,t}) - \hat{U}(t, K_{j-1,t})) \left. \right] dt \\
& \quad - \sum_{i=1}^{d_1} (L_{j,t}^{1i} - L_{j+1,t}^{1i})dW_t^{1i} - \int_Z (H_{j,t}^1(\theta) - H_{j+1,t}^1(\theta))\tilde{\mu}(d\theta, dt),
\end{aligned} \tag{5.32}$$

where we have used the equality (5.29) in Lemma 5.5.

Since $(\hat{U}(t, K_{j,t}) - \hat{U}(t, K_{j-1,t}))^* \hat{N}(t, K_{j,t})(\hat{U}(t, K_{j,t}) - \hat{U}(t, K_{j-1,t}))$ is nonnegative, according to Lemma 5.4, we conclude that $K_{j,t} - K_{j+1,t}$ is also nonnegative. This implies $\{K_j\}_{j=1}^\infty$ is a non-increasing sequence

$$CI \geq K_{1,t} \geq K_{2,t} \geq \cdots \geq K_{j,t} \geq \cdots \geq 0, \quad t \in [0, T].$$

It follows that $\{K_j\}$ converges almost surely to a nonnegative bounded, S^n -valued process K . According to Lebesgue's convergence theorem, we have

$$\lim_{j \rightarrow \infty} E \int_0^T |K_{j,t} - K_t|^q dt \rightarrow 0, \quad \forall q > 0. \tag{5.33}$$

Thus $\{K_j\}_{j=1}^\infty$ and $\{u(t, K_j)\}_{j=1}^\infty$ is a Cauchy sequence in the above sense. Again using Lebesgue's convergence theorem, for $t \in [0, T]$, we also have

$$\lim_{j \rightarrow \infty} E |K_{j,t} - K_t|^q \rightarrow 0, \quad \forall q > 0. \tag{5.34}$$

Applying Itô's formula to $|K_{k,t} - K_{j,t}|^2$, we get

$$\begin{aligned}
& E|K_{k,0} - K_{j,0}|^2 + \sum_{i=1}^{d_1} E \int_0^T |L_{k,t}^{1i} - L_{j,t}^{1i}|^2 dt + E \iint_{Z \times (0,T]} |H_{k,t}(\theta) - H_{j,t}(\theta)|^2 \nu(d\theta) dt \\
& = 2E \int_0^T \text{tr} \left[(K_{k,t} - K_{j,t}) \left(C_t^{1i*} (L_{k,t}^{1i} - L_{j,t}^{1i}) + (L_{k,t}^{1i} - L_{j,t}^{1i}) C_t^{1i} \right. \right. \\
& \quad + \int_Z E_t^*(\theta) (H_{k,t}(\theta) - H_{j,t}(\theta)) \nu(d\theta) + \int_Z (H_{k,t}(\theta) - H_{j,t}(\theta)) E_t(\theta) \nu(d\theta) \\
& \quad \left. \left. + \int_Z E_t^*(\theta) (H_{k,t}(\theta) - H_{j,t}(\theta)) E_t(\theta) \nu(d\theta) dt + R(j, k) \right. \right. \\
& \leq \frac{1}{2} \sum_{i=1}^{d_1} E \int_0^T |L_{k,t}^{1i} - L_{j,t}^{1i}|^2 dt + \frac{1}{2} E \iint_{Z \times (0,T]} |H_{k,t}(\theta) - H_{j,t}(\theta)|^2 \nu(d\theta) dt \\
& \quad + CE \int_0^T |K_{k,t} - K_{j,t}|^2 dt,
\end{aligned} \tag{5.35}$$

where

$$\begin{aligned}
R(j, k) = & 2 \int_0^T \text{tr} \left[(K_{k,t} - K_{j,t}) \left((K_{k,t} - K_{j,t}) A_t^* + A_t^* (K_{k,t} - K_{j,t}) \right. \right. \\
& + \sum_{i=1}^{d_2} C_t^{2i*} (K_{k,t} - K_{j,t}) C_t^{2i} + \hat{U}^*(t, K_{k-1,t}) \hat{N}(t, K_{k,t}) \hat{U}(t, K_{k-1,t}) \\
& - \hat{U}^*(t, K_{j-1,t}) \hat{N}(t, K_{j,t}) \hat{U}(t, K_{j-1,t}) - U^*(t, K_{k-1,t}) \hat{B}^*(t, K_{k,t}) \\
& \left. \left. - \hat{B}(t, K_{k,t}) U(t, K_{k-1,t}) + U^*(t, K_{j-1,t}) \hat{B}^*(t, K_{j,t}) + \hat{B}(t, K_{j,t}) U(t, K_{j-1,t}) \right) \right] dt
\end{aligned} \tag{5.36}$$

and C is some deterministic positive constant.

Thus from (5.33) and (5.34), we know that $\{L_j^1\}_{j=1}^\infty$ and $\{H_j\}_{j=1}^\infty$ is Cauchy sequences in $\mathcal{L}_{\mathcal{F}^*}^2((0, T]; (S^n)^{d_1})$ and $\mathcal{L}_{\mathcal{F}^*}^{\nu, 2}([0, T] \times Z; S^n)$ respectively. We denote the limits by L^1 and H respectively. In the end, passing the limit in Eq.(5.31), we obtain that (K, L^1, H) satisfies Eq.(5.30). Thus (K, L^1, H) is the solution of BSRDEJ (5.17).

Uniqueness

Suppose that BSRDEJ (5.17) has two solutions (K, L^1, H) and $(\bar{K}, \bar{L}^1, \bar{H})$. Then it follows from (5.30) that

$$\begin{aligned}
-dK_t = & \left[F(t, K_t, L_t^1, H_t, \hat{U}(t, K_t)) + \hat{U}^*(t, K_t) N_t \hat{U}(t, K_t) + Q_t \right] dt \\
& - \sum_{i=1}^{d_1} L_t^{1i} dW_t^{1i} - \int_Z H_t(\theta) \tilde{\mu}(d\theta, dt), \quad K_T = M
\end{aligned}$$

and

$$\begin{aligned}
-d\bar{K}_t = & \left[F(t, \bar{K}_t, \bar{L}_t^1, \bar{H}_t, \hat{U}(t, \bar{K}_t)) + \hat{U}^*(t, \bar{K}_t) N_t \hat{U}(t, \bar{K}_t) + Q_t \right] dt \\
& - \sum_{i=1}^{d_1} \bar{L}_t^{1i} dW_t^{1i} - \int_Z \bar{H}_t(\theta) \tilde{\mu}(d\theta, dt), \quad \bar{K}_T = M,
\end{aligned}$$

Thus

$$\begin{aligned}
-d(\bar{K}_t - K_t) &= \left[F(t, \bar{K}_t - K_t, \bar{L}_t^1 - L_t^1, \bar{H}_t - H_t, \hat{U}(t, \bar{K}_t)) \right. \\
&\quad + F(t, K_t, L_t^1, H_t, \hat{U}(t, \bar{K}_t)) + \hat{U}^*(t, \bar{K}_t) N_t \hat{U}(t, \bar{K}_t) \\
&\quad \left. - F(t, K_t, L_t^1, H_t, \hat{U}(t, K_t)) - \hat{U}^*(t, K_t) N_t \hat{U}(t, K_t) \right] dt \\
&\quad - \sum_{i=1}^{d_1} (\bar{L}_t^{1i} - L_t^{1i}) dW_t^{1i} - \int_Z (\bar{H}_t(\theta) - H_t(\theta)) \tilde{\mu}(d\theta, dt) \\
&= \left[F(t, \bar{K}_t - K_t, \bar{L}_t^1 - L_t^1, \bar{H}_t - H_t, \hat{U}(t, \bar{K}_t)) \right. \\
&\quad + (\hat{U}(t, K_t) - \hat{U}(t, \bar{K}_t))^* \hat{N}(t, K_t) (\hat{U}(t, K_t) - \hat{U}(t, \bar{K}_t)) \left. \right] dt \\
&\quad - \sum_{i=1}^{d_1} (\bar{L}_t^{1i} - L_t^{1i}) dW_t^{1i} - \int_Z (\bar{H}_t(\theta) - H_t(\theta)) \tilde{\mu}(d\theta, dt), \\
\bar{K}_T - K_T &= 0.
\end{aligned} \tag{5.37}$$

Since $(\hat{U}(t, K_t) - \hat{U}(t, \bar{K}_t))^* \hat{N}(t, K_t) (\hat{U}(t, K_t) - \hat{U}(t, \bar{K}_t))$ is nonnegative, it follows from Lemma 5.4 that $\bar{K} - K$ is also a.e. a.s. nonnegative. Similarly we can obtain that $\bar{K} - K$ is a.e.a.s. nonnegative. This implies $K = \bar{K}$. In the end, from the uniqueness result of Lemma 5.4, we conclude that $\bar{L}^1 = L^1, \bar{H} = H$. The uniqueness is proved. \square

Acknowledgements The author is very grateful to Professor Tang shanjian for his valuable suggestions and various instruction.

References

- [1] R. Bellman. Functional equations in the theory of dynamic programming, positivity and quasilinearity. *Proc. Natl. Acad. Sci. USA*, 41:743–746, 1955.
- [2] R. E. Bellman, I. L. Glicksberg, and O. L. Gross. *Some aspects of the mathematical theory of control processes*. Rand Co., Santa Monica-California, 1958.
- [3] J. M. Bismut. Linear quadratic optimal stochastic control with random coefficients, 14 (1976), pp. 419c444. *SIAM J. Control Optim.*, 14:419–444, 1976.
- [4] S. Chen, X. Li, and X. Zhou. Stochastic linear quadratic regulators with indefinite control weight costs,. *SIAM J. Control Optim.*, 36:1685–1702, 1998.
- [5] S. Chen and S. Tang. Semi-linear backward stochastic integral partial differential equations driven by a brownian motion and a poisson point process. *arxiv.org/abs/1007.3201*.
- [6] S. Chen and J. Yong. Stochastic linear quadratic optimal control problems,. *SIAM J. Control Optim.*, 39:21–45, 2001.

- [7] S. Chen and X. Zhou. Stochastic linear quadratic regulators with indefinite control weight costs. ii. *SIAM J. Control Optim.*, 39:1065–1081, 2000.
- [8] I. Ekeland and R. Témam. *Convex analysis and variational problems*. Amsterdam: North-Holland, 1976.
- [9] Y. Hu and B. Økandal. Partial information linear quadratic control for jump diffusions. *SIAM J. Control Optim.*, 47:1744–1761, 2008.
- [10] Y. Hu and X. Zhou. Constrained stochastic lq control with random coefficients, and application to mean–variance portfolio selection. *SIAM J. Control Optim.*, 44:444–466, 2005.
- [11] R.E. Kalman. Contributions to the theory of optimal control. *Bol. Soc. Mat. Mexicana*, 5:102–119, 1960.
- [12] M. Kohlmann and S. Tang. Minimization of risk and linear quadratic optimal control theory. *SIAM J. Control Optim.*, 42:1118–1142, 2003.
- [13] M. Kohlmann and S. Tang. Multidimensional backward stochastic riccati equations and applications. *SIAM J. Control Optim.*, 41:1696–1721, 2003.
- [14] M. Kohlmann and X. Zhou. Relationship between backward stochastic differential equations and stochastic controls: A linear-quadratic approach. *SIAM J. Control Optim.*, 38:1392–1407, 2000.
- [15] Q. Meng and S. Tang. Backward stochastic HJB equations with jumps, preprint.
- [16] S. Peng. Stochastic Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.*, 30:284–304, 1992.
- [17] M. A. Rami and X. Zhou. Linear matrix inequalities, riccati equations, and indefinite stochastic linear quadratic controls. *IEEE Trans. Automat. Control*, 45:1131–1143, 2000.
- [18] S. Tang. General linear quadratic optimal stochastic control problems with random coefficients: linear stochastic hamilton systems and backward stochastic riccati equations. *SIAM J. Control Optim.*, 42:53–75, 2003.
- [19] S. Tang and X. Li. Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM J. Control Optim.*, 32:1447–1475, 1994.
- [20] W.M. Wonham. On a matrix riccati equation of stochastic control. *SIAM J. Control Optim.*, 6:312–326, 1968.
- [21] Z. Wu and X. Wang. FBSDE with poisson process and its application to linear quadratic stochastic optimal control problem with random jumps. *Acta Automatica Sinica*, 29:821–826, 2003.
- [22] D. Yao, S.Zhang, and X. Zhou. Stochastic LQ control via primal–dual semidefinite programming. *SIAM Review*, 46:87–111, 2004.